Statistics & Risk Modeling 28, 299–317 (2011) / **DOI** 10.1524/strm.2011.1096

Risk margin for a non-life insurance run-off

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Received: February 15, 2011; Accepted: September 9, 2011

Summary: For solvency purposes insurance companies need to calculate so-called best-estimate reserves for outstanding loss liability cash flows and a corresponding risk margin for non-hedgeable insurance-technical risks in these cash flows. In actuarial practice, the calculation of the risk margin is often not based on a sound model but various simplified methods are used. In the present paper we properly define these notions and we introduce insurance-technical probability distortions. We describe how the latter can be used to calculate a risk margin for non-life insurance run-off liabilities in a mathematically consistent way.

1 Market-consistent valuation

The main task of an actuary is to predict and value insurance cash flows. These predictions and valuations form the basis for premium calculations as well as for solvency considerations of an insurance company. As a consequence, and in order to be able to successfully run the insurance business, actuaries need to have a good understanding of such insurance cash flows. In most situations, insurance cash flows are not traded on deep and liquid financial markets. Therefore valuation of insurance cash flows basically means pricing in an incomplete financial market setting. Article 75 of the Solvency II Framework Directive (Directive 2009/138/EC) states “liabilities shall be valued at the amount for which they could be transferred, or settled, between two knowledgeable willing parties in an arm’s length transaction”. The general understanding is that this amount should consist of two components, namely the so-called best-estimate reserves for the cash flows and a risk margin for non-hedgeable risks in these cash flows. We will discuss these two elements in detail by giving an economically based approach how they can be calculated.

The calculation of the *best-estimate reserves* is fairly straightforward. Article 77 of the Solvency II Framework Directive says “the best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of time value of money ... the calculation of the best estimate shall be based upon up-to-date and credible

AMS 2000 subject classification: Primary: 62P05; Secondary: 91G99
Key words and phrases: Claims reserving, best-estimate reserves, run-off risks, risk margin, market value margin, one-year uncertainty, claims development result, market-consistent valuation

information ... ". This simply means that the best-estimate reserves are a time value adjusted conditional expectation of future cash flows, conditioned on the information that we have collected up to today.

The calculation of the *risk margin* has led to more discussion as there is no general understanding on how it should be calculated. The most commonly used approach is the so-called *cost-of-capital approach*. The cost-of-capital approach is based on the reasoning that a financial agent provides for every future accounting year the risk bearing capital that protects against adverse developments in the run-off of the insurance cash flows. Since that financial agent provides this yearly protection, a reward in the form of a yearly price is expected. The total of these yearly prices constitutes the so-called cost-of-capital margin which is then set equal to the risk margin. The difficulty with this cost-of-capital approach is that in almost all situations it is not tractable. It involves path-dependent multi-period risk measures; see Salzmann and Wüthrich [10]. In most interesting cases these path-dependent multi-period risk measure loadings can not be calculated analytically, nor can they be calculated numerically in an efficient way because they usually involve large amounts of nested simulations. Therefore, various proxies are used in practice. Probably the two most commonly used proxies are the proportional scaling proxy and the split of total uncertainty proxy; see Salzmann and Wüthrich [10], Wüthrich [13] and Articles TP.5.32 and TP.5.41 of QIS5 [9]. Related papers are Artzner and Eisele [1] and Möhr [8].

In this paper we present a completely different, more economically based approach. We argue that the risk margin should be related to the risk aversion of the financial agent that provides the protection against adverse developments. This risk aversion can be modeled using probability distortion techniques and this will lead to a mathematically fully consistent risk margin. Under the proposed method, risk-adjusted values of insurance cash flows are calculated as expected values after modifying (distorting) the probabilities used. This kind of idea has been used in actuarial practice for a very long time, however typically in the field of life insurance mathematics, corresponding to the construction of first order life tables out of second order life tables. Second order life tables are expected death/survival probabilities whereas for first order life tables a safety loading is added to insure that the (life) insurance premium is sufficiently high.

We apply these ideas to the context of non-life insurance liabilities. We study the run-off of outstanding loss liabilities in a chain ladder framework. Using probability distortions, we develop so-called risk-adjusted chain ladder factors from the classical chain ladder factors. These risk-adjusted factors have a surprisingly simple form and allow for a natural inclusion of the risk margin into our considerations. Related literature to these probability distortion considerations (and the related change-of-measure techniques in financial mathematics) are, among others, Bühlmann et al. [2], Denuit et al. [4], Föllmer and Schied [5], Tsanakas and Christofides [11], Wang [12] and Wüthrich et al. [14].

The paper is organized as follows. In the next section we define the Bayesian log-normal chain ladder model for claims reserving. Within this model we then calculate the best-estimate reserves as required by the solvency directive; see Section 3 below. In Section 4 we introduce general insurance-technical probability distortions. An explicit choice for the latter then provides the positive risk margin. Finally, in Section 5 we provide a real data example that is based on private liability insurance data. We compare our numerical results to other concepts used in practice. All the proofs of the statements are provided in the Appendix.

2 Model assumptions

We assume that a final time horizon $n \in \mathbb{N}$ is given and consider the insurance cash flow valuation problem in discrete time $t \in \{0, \ldots, n\}$. For simplicity we assume that the time unit corresponds to years. We denote the underlying probability space by $(\Omega, \mathcal{G}, \mathbb{P})$ and assume that, on this probability space, we have two flows of information given by the filtrations $\mathbb{F} = (\mathcal{F}_t)_{t=0,\ldots,n}$ and $\mathbb{T} = (\mathcal{T}_t)_{t=0,\ldots,n}$. We assume $\mathcal{F}_0$ and $\mathcal{T}_0$ are the trivial σ-fields. The filtration $\mathbb{F}$ corresponds to the financial market filtration and $\mathbb{T}$ corresponds to the insurance-technical filtration. In order to keep the model simple, we assume that these two filtrations are stochastically independent under the probability law $\mathbb{P}$; see also Section 2.6 in Wüthrich et al. [14]. Of course, this last assumption can be rather restrictive in applications, however, we emphasize that it can be relaxed by expressing insurance liabilities in the right financial units; see the valuation portfolio construction in Wüthrich et al. [14].

This independent decoupling into financial variables adapted to $\mathbb{F}$ and insurance-technical variables adapted to $\mathbb{T}$ implies that we can replicate *expected* insurance cash flows in terms of default-free zero coupon bonds; see Assumption 5.1 and Remark 5.2 in Wüthrich et al. [14]. This is in-line with Article 77 of the Solvency II Framework Directive, but needs to be questioned if we have no independent decoupling into financial and insurance-technical variables.

Insurance cash flows are denoted by $X_{i,j}$, where $i \in \{1, \ldots, I\}$ are the accident years of the insurance claims (origin years) and $j \in \{0, \ldots, J\}$ are the development years of these insurance claims (payment delays). We assume that all claims are settled after development year J and that $I \geq J + 1$. With this terminology, cash flow $X_{i,j}$ is paid in accounting year $k = i + j$. This provides the accounting year cash flows (over all accident years $i \in \{1, \ldots, I\}$)

$$X_k = \sum_{i+j=k} X_{i,j} = \sum_{i=1\vee(k-J)}^{I\wedge k} X_{i,k-i} = \sum_{j=0\vee(k-I)}^{J\wedge(k-1)} X_{k-j,j}.$$

We denote the total cash flow by $\mathbf{X} = (X_1, \ldots, X_n)$ and the outstanding loss liabilities at time $t < n$ are given by

$$\mathbf{X}_{(t+1)} = (0, \ldots, 0, X_{t+1}, \ldots, X_n).$$

Thus, our aim is to model, predict and value this outstanding loss liability cash flow $\mathbf{X}_{(t+1)}$ for every $t < n$. For the modeling of the cash flow $\mathbf{X}$ we use the following Bayesian chain ladder model.

Model 2.1 (Bayesian log-normal chain ladder model) *We assume $n = I + J$ and*

- $\mathcal{T}_t = \sigma \left\{ X_{i,j};\, i + j \leq t, i = 1, \ldots, I, j = 0, \ldots, J \right\}$ *for all* $t = 1, \ldots, I + J$*;*
- *conditionally, given* $\boldsymbol{\Phi} = (\Phi_0, \ldots, \Phi_{J-1})$ *and* $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{J-1})$*, we have*

- $X_{i,j}$ *are independent for different accident years* i*;*
- *cumulative payments* $C_{i,j} = \sum_{l=0}^{j} X_{i,l}$ *satisfy*

$$\xi_{i,j+1} \stackrel{\text{def.}}{=} \log\left(\frac{C_{i,j+1}}{C_{i,j}} - 1\right)\Bigg|_{\mathcal{T}_{i+j},\boldsymbol{\Phi},\boldsymbol{\sigma}} \sim \mathcal{N}\left(\Phi_j, \sigma_j^2\right)$$

for $j = 0, \ldots, J-1$ *and* $i = 1, \ldots, I$*;*

- $\boldsymbol{\sigma} > 0$ *is deterministic and* Φ_j*,* $j = 0, \ldots, J-1$*, are independent with*

$$\Phi_j \sim \mathcal{N}\left(\phi_j, s_j^2\right),$$

with prior parameters ϕ_j *and* $s_j > 0$*, and*

- $(X_{1,0}, \ldots, X_{I,0})$ *and* $\boldsymbol{\Phi}$ *are independent.*

We assume that the insurance-technical filtration $\mathbb{T}$ is generated by the insurance cash flows $X_{i,j}$. This suggests that this is the only insurance-technical information available to solve the cash flow prediction problem. Moreover, since we have assumed independence between $\mathbb{F}$ and $\mathbb{T}$ we know that the time value adjustments of cash flows need to be done with default-free zero coupon bonds. This immediately implies that the best-estimate reserves for the outstanding loss liabilities at time $t < n$ are given by

$$\mathcal{R}_t\left(\mathbf{X}_{(t+1)}\right) = \sum_{k \geq t+1} \mathbb{E}\left[X_k \middle| \mathcal{T}_t\right] P(t,k) = \sum_{k \geq t+1} \sum_{i+j=k} \mathbb{E}\left[X_{i,j} \middle| \mathcal{T}_t\right] P(t,k), \tag{2.1}$$

where $P(t, k)$ is the price at time t of the default-free zero coupon bond that matures at time k. This definition of best-estimate reserves provides the necessary martingale framework for the joint filtration of $\mathbb{F}$ and $\mathbb{T}$ (under the measure $\mathbb{P}$) which in these terms provides an arbitrage-free pricing framework; for more details see Chapter 2 in Wüthrich et al. [14].

We have chosen a Bayesian Ansatz in the assumptions of Model 2.1. The advantage of a Bayesian model is that the parameter uncertainty is, in a natural way, included in the model, and parameter estimation is canonical using posterior distributions. Moreover, we have chosen an exact credibility model (see Bühlmann and Gisler [3, Chapter 2]) which has the advantage that we obtain closed form solutions for posterior distributions. However, our analysis is by no means restricted to the Bayesian log-normal chain ladder model. Other models can be solved completely analogously, but in some cases one has to rely on simulation methods such as the Markov Chain Monte Carlo (MCMC) simulation methodology.

3 Best-estimate reserves calculation

In formula (2.1) we have defined the best-estimate reserves. In this section we calculate these best-estimate reserves explicitly for Model 2.1. We assume that $t \geq I$, which implies that at time t all initial payments $X_{i,0}$ have been observed for accident years

$i \in \{1, \ldots, I\}$. For $i + j > t$ we then obtain, using the tower property for conditional expectations (note that we also condition on the model parameters $\mathbf{\Phi}$),

$$\mathbb{E}\left[X_{i,j} \middle| \mathcal{T}_t, \mathbf{\Phi} \right] = C_{i,t-i} \left(\prod_{l=t-i}^{j-2} \left(\exp\left\{ \Phi_l + \sigma_l^2/2 \right\} + 1 \right) \right) \exp\left\{ \Phi_{j-1} + \sigma_{j-1}^2/2 \right\}. \quad (3.1)$$

For a proof, we refer to Lemma 5.2 in Wüthrich and Merz [15]. Formula (3.1) implies that we would like to do Bayesian inference on $\mathbf{\Phi}$, given the observations $\mathcal{T}_t$. That is, we would like to determine the posterior distribution of $\mathbf{\Phi}$ at time t. This then provides the Bayesian predictor

$$\mathbb{E}\left[X_{i,j} \middle| \mathcal{T}_t \right] = C_{i,t-i} \mathbb{E}\left[\left(\prod_{l=t-i}^{j-2} \left(\exp\left\{ \Phi_l + \sigma_l^2/2 \right\} + 1 \right) \right) \exp\left\{ \Phi_{j-1} + \sigma_{j-1}^2/2 \right\} \middle| \mathcal{T}_t \right].$$

In Model 2.1 we can explicitly provide the posterior density of $\mathbf{\Phi}$, given the observations $\mathcal{T}_t$:

$$h\left(\mathbf{\Phi} \middle| \mathcal{T}_t \right) \propto \prod_{j=0}^{J-1} \exp\left\{ -\frac{1}{2s_j^2} \left(\Phi_j - \phi_j \right)^2 \right\} \prod_{i=1}^{I} \prod_{j=1}^{(t-i)\wedge J} \exp\left\{ -\frac{1}{2\sigma_{j-1}^2} \left(\xi_{i,j} - \Phi_{j-1} \right)^2 \right\}.$$

The first term on the right-hand side is the prior information about the parameters $\mathbf{\Phi}$, the second term is the likelihood function of the observations, given the parameters $\mathbf{\Phi}$. This posterior density immediately provides the following theorem.

Theorem 3.1 *In Model 2.1, the posteriors of Φ_j, given $\mathcal{T}_t$ with $t \geq I$, are independent normally distributed random variables with*

$$\Phi_j \big|_{\mathcal{T}_t} \sim \mathcal{N}\left(\phi_j^{(t)}, \left(s_j^{(t)} \right)^2 \right),$$

and posterior parameters

$$\phi_j^{(t)} = \left(s_j^{(t)} \right)^2 \left[\frac{\phi_j}{s_j^2} + \frac{1}{\sigma_j^2} \sum_{i=1}^{(t-j-1)\wedge I} \xi_{i,j+1} \right]$$

and

$$\left(s_j^{(t)} \right)^2 = \left(\frac{1}{s_j^2} + \frac{(t-j-1)\wedge I}{\sigma_j^2} \right)^{-1}.$$

Theorem 3.1 implies that

$$\phi_j^{(t)} = \mathbb{E}\left[\Phi_j \middle| \mathcal{T}_t \right] = \beta_j^{(t)} \overline{\xi}_j^{(t)} + \left(1 - \beta_j^{(t)} \right) \phi_j, \quad (3.2)$$

with sample mean and credibility weight given by

$$\overline{\xi}_j^{(t)} = \frac{1}{(t-j-1)\wedge I} \sum_{i=1}^{(t-j-1)\wedge I} \xi_{i,j+1} \qquad \text{and} \qquad \beta_j^{(t)} = \frac{\left[(t-j-1)\wedge I \right] s_j^2}{\sigma_j^2 + \left[(t-j-1)\wedge I \right] s_j^2}.$$

Hence, the posterior mean of Φ_j is a credibility weighted average between the sample mean $\overline{\xi}_j^{(t)}$ and the prior mean ϕ_j with credibility weight $\beta_j^{(t)}$. For non-informative prior information we let $s_j \to \infty$ and find that $\beta_j^{(t)} \to 1$ which means that we give full credibility to the observation based parameter estimate $\overline{\xi}_j^{(t)}$. For perfect prior information we let $s_j \to 0$ and conclude that $\beta_j^{(t)} \to 0$, i.e. we give full credibility to the prior estimate ϕ_j.

Using the posterior independence and Gaussian properties of Φ_j we obtain the following corollary for the Bayesian predictor.

Corollary 3.2 *In Model 2.1 we obtain, for $i + j > t \geq I$,*

$$\mathbb{E}\left[X_{i,j} \middle| \mathcal{T}_t \right] = C_{i,t-i} \left(\prod_{l=t-i}^{j-2} f_l^{(t)} \right) \left(f_{j-1}^{(t)} - 1 \right),$$

with posterior chain ladder factors

$$f_l^{(t)} = \mathbb{E}\left[\exp\left\{ \Phi_l + \sigma_l^2/2 \right\} + 1 \middle| \mathcal{T}_t \right] = \exp\left\{ \phi_l^{(t)} + \left(s_l^{(t)}\right)^2/2 + \sigma_l^2/2 \right\} + 1.$$

Moreover, $(f_l^{(t)})_{t=0,\ldots,n}$ are $(\mathbb{P}, \mathbb{T})$-martingales for all $l = 0, \ldots, J-1$.

This corollary has the consequence that, in Model 2.1, the best-estimate reserves at time $t \geq I$ are given by

$$\mathcal{R}_t \left(\mathbf{X}_{(t+1)} \right) = \sum_{i=t+1-J}^{I} C_{i,t-i} \sum_{j=t-i+1}^{J} \left(\prod_{l=t-i}^{j-2} f_l^{(t)} \right) \left(f_{j-1}^{(t)} - 1 \right) P(t, i+j). \tag{3.3}$$

This solves the question about the calculation of best-estimate reserves for outstanding loss liabilities: it is a probability-weighted, time value adjusted amount that considers the most recent available information. We now turn to the more challenging calculation of the risk margin which covers deviations from these best-estimate reserves.

4 Risk-adjusted reserves and risk margin

In this section we define the risk margin using the economic argument that a risk averse financial agent will ask for a premium that is higher than the conditionally expected discounted claim. This will be achieved by introducing a probability distortion on the payments $X_{i,j}$ which will lead to the so-called risk-adjusted reserves $\mathcal{R}_t^+ \left(\mathbf{X}_{(t+1)} \right)$ at time t. The risk margin at time t can then by defined as the difference

$$\mathrm{RM}_t \left(\mathbf{X}_{(t+1)} \right) = \mathcal{R}_t^+ \left(\mathbf{X}_{(t+1)} \right) - \mathcal{R}_t \left(\mathbf{X}_{(t+1)} \right), \tag{4.1}$$

which will be strictly positive under an appropriate probability distortion. Before doing this explicitly for the Bayesian chain ladder model, we describe the probability distortions that we are going to use in more generality. The crucial idea is that we introduce a density process $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n)$ that modifies the probabilities in an appropriate way. The probability distortion functions introduced by Wang [12] relate to our framework in sufficiently smooth cases and the change-of-measure techniques from financial mathematics are obtained by the transformations presented in Sections 2.5 and 2.6 of Wüthrich et al. [14].

4.1 Insurance-technical probability distortions

An insurance-technical probability distortion $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n)$ is a $\mathbb{T}$-adapted and strictly positive stochastic process that is a $(\mathbb{P}, \mathbb{T})$-martingale with normalization $\varphi_0 = 1$. This is exactly the definition given in (2.103) of Wüthrich et al. [14] and means that $\boldsymbol{\varphi}$ is a density process w.r.t. $(\mathbb{P}, \mathbb{T})$ (which can be used for a change-of-measure). For a cash flow $\mathbf{X}$ we can then define the risk-adjusted units by

$$\Lambda_{t,k} = \frac{1}{\varphi_t} \mathbb{E}\left[\left. \varphi_k X_k \right| \mathcal{T}_t\right].$$

In view of (2.1), the risk-adjusted reserves are then defined by

$$\mathcal{R}_t^+\left(\mathbf{X}_{(t+1)}\right) = \sum_{k \geq t+1} \Lambda_{t,k} P(t,k) = \sum_{k \geq t+1} \sum_{i+j=k} \frac{1}{\varphi_t} \mathbb{E}\left[\left. \varphi_k X_{i,j} \right| \mathcal{T}_t\right] P(t,k). \tag{4.2}$$

For the choice $\boldsymbol{\varphi} \equiv 1$ the best-estimate reserves and the risk-adjusted reserves coincide, but for an appropriate risk averse choice of $\boldsymbol{\varphi}$ we will obtain a strictly positive risk margin $\mathrm{RM}_t\left(\mathbf{X}_{(t+1)}\right)$.

For the latter, it is required that $\varphi_k | \mathcal{T}_t$ and $X_k | \mathcal{T}_t$ are positively correlated, where in this case (using the martingale property of $\boldsymbol{\varphi}$)

$$\Lambda_{t,k} = \frac{1}{\varphi_t} \mathbb{E}\left[\left. \varphi_k X_k \right| \mathcal{T}_t\right] \geq \frac{1}{\varphi_t} \mathbb{E}\left[\left. \varphi_k \right| \mathcal{T}_t\right] \mathbb{E}\left[\left. X_k \right| \mathcal{T}_t\right] = \mathbb{E}\left[\left. X_k \right| \mathcal{T}_t\right].$$

This correlation inequality is often achieved by using the Fortuin–Kasteleyn–Ginibre (FKG) inequality from [6], which sometimes is also called the supermodular property. The positive correlatedness implies that more probability weight is given to adverse scenarios. In order to have time-consistency w.r.t. to risk aversion, we require that $(\Lambda_{t,k})_{t=0,\ldots,n}$ is a $(\mathbb{P}, \mathbb{T})$ super-martingale. This implies that

$$\mathbb{E}\left[\left. \Lambda_{t+1,k} - \mathbb{E}\left[\left. X_k \right| \mathcal{T}_{t+1}\right] \right| \mathcal{T}_t\right] \leq \Lambda_{t,k} - \mathbb{E}\left[\left. X_k \right| \mathcal{T}_t\right], \tag{4.3}$$

which says that, in expectation, the risk margin is constantly released over time.

4.2 Risk-adjusted reserves for the Bayesian chain ladder model

In the previous section, using insurance-technical probability distortions, we have given the general concept for the calculation of a positive risk margin. In the present section we give an explicit example for the insurance-technical probability distortion $\boldsymbol{\varphi}$ that will fit to our Bayesian chain ladder model. We make the following particular choice:

$$\varphi_n = \prod_{j=1}^{J} \exp\left\{\alpha_1 \sum_{i=1}^{I} \xi_{i,j} + \alpha_2 \Phi_{j-1} - (I\alpha_1 + \alpha_2)\phi_{j-1} - (I\alpha_1 + \alpha_2)^2 \frac{s_{j-1}^2}{2} - I\alpha_1^2 \frac{\sigma_{j-1}^2}{2}\right\}, \tag{4.4}$$

where $\alpha_1, \alpha_2 \geq 0$ are fixed constants. As will become apparent below, the parameters α_1 and α_2 characterize risk aversion: α_1 relates to the process risk in $\xi_{i,j}$ and α_2 to the parameter uncertainty in $\boldsymbol{\Phi}$. We then define the insurance-technical probability distortion $\boldsymbol{\varphi}$ by $\varphi_t = \mathbb{E}\left[\varphi_n \middle| \mathcal{T}_t\right]$.

Lemma 4.1 *$\boldsymbol{\varphi}$ is a strictly positive and normalized* $(\mathbb{P}, \mathbb{T})$*-martingale.*

The proof of the lemma is provided in the Appendix. We are now ready to state our main theorem.

Theorem 4.2 *In Model 2.1 we have, for $k > t \geq I$ and $i \in \{k - J, \ldots, I\}$,*

$$\frac{1}{\varphi_t} \mathbb{E}\left[\varphi_k X_{i,k-i} \middle| \mathcal{T}_t\right] = C_{i,t-i} \left(\prod_{l=t-i}^{k-i-2} f_l^{(+t)}\right) \left(f_{k-i-1}^{(+t)} - 1\right),$$

with risk-adjusted chain ladder factors

$$f_l^{(+t)} = \exp\left\{\phi_l^{(t)} + \frac{\left(s_l^{(t)}\right)^2}{2} + \frac{\sigma_l^2}{2}\right\} \times \exp\left\{(\alpha_2 + [I - (t - l - 1)]\alpha_1) \left(s_l^{(t)}\right)^2 + \alpha_1 \sigma_l^2\right\} + 1.$$

The theorem is proved in the Appendix. In view of Corollary 3.2 and Theorem 4.2 we obtain, for $l \geq t - I$, the inequality $f_l^{(+t)} \geq f_l^{(t)}$. The posterior chain ladder factors $f_l^{(t)}$ provide the best-estimate reserves at time t, the risk-adjusted chain ladder factors $f_l^{(+t)}$ provide risk-adjusted reserves that consider both process risk in $\xi_{i,j}$ and parameter uncertainty in Φ_j. The risk-adjusted reserves are then given by

$$\mathcal{R}_t^+ \left(\mathbf{X}_{(t+1)}\right) = \sum_{i=t+1-J}^{I} C_{i,t-i} \sum_{j=t-i+1}^{J} \left(\prod_{l=t-i}^{j-2} f_l^{(+t)}\right) \left(f_{j-1}^{(+t)} - 1\right) P(t, i + j), \tag{4.5}$$

and we obtain a positive risk margin $\mathrm{RM}_t \left(\mathbf{X}_{(t+1)}\right)$.

Remark 4.3 • We observe that it is fairly easy to calculate the risk-adjusted reserves in the Bayesian log-normal chain ladder Model 2.1 with probability distortion (4.4), all that we need to do is to modify the chain ladder factors appropriately:

$$f_l^{(+t)} = \left(f_l^{(t)} - 1\right) \exp\left\{(\alpha_2 + [I - (t - l - 1)]\alpha_1) \left(s_l^{(t)}\right)^2 + \alpha_1 \sigma_l^2\right\} + 1. \tag{4.6}$$

The following function for $l \geq t - I \geq 0$,

$$\tau_{l,t}(\alpha_1, \alpha_2) = \exp\left\{(\alpha_2 + [I - (t - l - 1)]\alpha_1)\left(s_l^{(t)}\right)^2 + \alpha_1 \sigma_l^2\right\} \geq 1$$

exactly reflects this modification according to the risk aversion parameters $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. Note that $\tau_{l,t}(\alpha_1, \alpha_2)$ is deterministic and, as stated before, represents the level of prudence similar to the construction of the first and second order life tables in life insurance.

- The parameter α_2 reflects risk aversion in the parameter uncertainty and the parameter α_1 reflects risk aversion in the process risk. However, α_1 also influences parameter uncertainty because in the Bayesian analysis we do inference on the parameters from the observed information $\mathcal{T}_t$.

- This concept of constructing risk-adjusted chain ladder factors is by no means exclusive to the Bayesian log-normal chain ladder model. It can be applied to other chain ladder models, or even more broadly, to every claims reserving and pricing model (similar as the risk neutral measure constructions in financial mathematics). It hence yields a very general concept for constructing a risk margin. We have chosen the Bayesian log-normal chain ladder model because of its practical relevance and because it allows for closed form solutions, helping interpretation. Note that (4.4) gives a special type of probability distortion, other choices could have been made. The remaining, more economic and regulatory, question then is: which are alternative constructions of insurance-technical probability distortions used in practice, and how should these be calibrated?

4.3 Expected run-off of the risk margin

In this subsection we study the expected run-off of the best-estimate and of the risk-adjusted reserves. For this, we need the following lemma.

Lemma 4.4 *For $l \geq t - I \geq s - I \geq 0$ we have*

$$f_l^{(+t,s)} = \mathbb{E}\left[\left. f_l^{(+t)} \right| \mathcal{T}_s\right] = \left(f_l^{(s)} - 1\right)\tau_{l,t}(\alpha_1, \alpha_2) + 1.$$

The proof of this lemma immediately follows from (4.6) and the martingale property of the chain ladder factors $(f_l^{(t)})_{t=0,\ldots,n}$. Observe that $\tau_{l,t}(\alpha_1, \alpha_2)$ is decreasing in t which gives the super-martingale property (4.3). Moreover, we have the following theorem.

Theorem 4.5 *For $t > s \geq I$ we have for the expected best-estimate reserves*

$$\mathbb{E}\left[\mathcal{R}_t\left(\mathbf{X}_{(t+1)}\right)\middle| \mathcal{T}_s, \mathcal{F}_s\right] = \sum_{i=t+1-J}^{I} C_{i,s-i} \sum_{j=t-i+1}^{J} \prod_{l=s-i}^{j-2} f_l^{(s)} \left(f_{j-1}^{(s)} - 1\right) \mathbb{E}\left[P(t, i + j)\middle| \mathcal{F}_s\right],$$

and for the expected risk-adjusted reserves

$$\mathbb{E}\left[\left.\mathcal{R}_t^+\left(\mathbf{X}_{(t+1)}\right)\right|\mathcal{T}_s,\mathcal{F}_s\right]$$

$$= \sum_{i=t+1-J}^{I}\left[C_{i,s-i}\prod_{l=s-i}^{t-i-1} f_l^{(s)}\right.$$

$$\left.\times \sum_{j=t-i+1}^{J}\prod_{l=t-i}^{j-2} f_l^{(+t,s)}\left(f_{j-1}^{(+t,s)}-1\right)\mathbb{E}\left[\left.P(t,i+j)\right|\mathcal{F}_s\right]\right].$$

Note that, in order to project the expected run-off of the best-estimate reserves and the risk margin for $t \geq s \geq I$, we also need to model the expected future zero coupon bond prices $\mathbb{E}\left[\left.P(t,i+j)\right|\mathcal{F}_s\right]$. In the next section we give a numerical example for this run-off.

5 Real data example

We present a real data example. The data set is a 17×17 private liability insurance cash flow triangle. In Table 5.1 we provide the cumulative payments $C_{i,j} = \sum_{l=0}^{j} X_{i,l}$ for $i + j \leq 17$. We choose the final accident year under consideration $I = 17$ and we assume that all claims are settled after development year $J = 16$. We then consider the run-off situation at time I for $t = I, \ldots, n = 33$.

Using the parameter choices from Table 5.1 we are able to calculate the credibility weights $\beta_j^{(t)}$ and the posterior means $\phi_j^{(t)}$ at time $t = 17$.

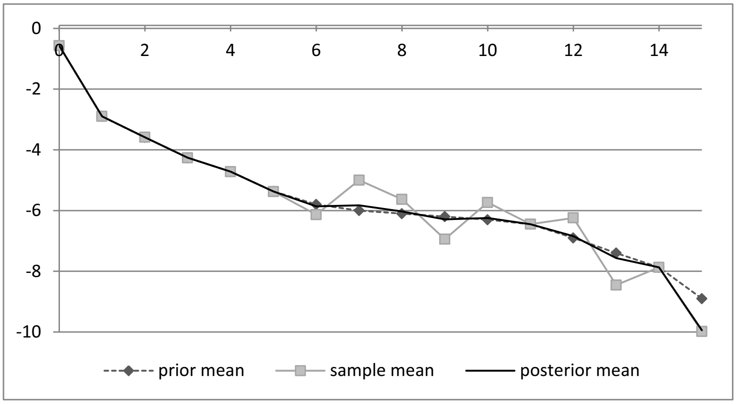

Figure 5.1 Prior mean ϕ_j, sample mean $\overline{\xi}_j^{(t)}$ and posterior mean $\phi_j^{(t)}$ for $j = 0, \ldots, 15$ and $t = 17$.

a.y.	development year j																
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	13'109	20'355	21'337	22'043	22'401	22'658	22'997	23'158	23'492	23'664	23'699	23'904	23'960	23'992	23'994	24'001	24'002
2	14'457	22'038	22'627	23'114	23'238	23'312	23'440	23'490	23'964	23'976	24'048	24'111	24'252	24'538	24'540	24'550	
3	16'075	22'672	23'753	24'052	24'206	24'757	24'786	24'807	24'823	24'888	24'986	25'401	25'681	25'705	25'732		
4	15'682	23'464	24'465	25'052	25'529	25'708	25'752	25'770	25'835	26'075	26'082	26'146	26'150	26'167			
5	16'551	23'706	24'627	25'573	26'046	26'115	26'283	26'481	26'701	26'718	26'724	26'728	26'735				
6	15'439	23'796	24'866	25'317	26'139	26'154	26'175	26'205	26'764	26'818	26'836	26'959					
7	14'629	21'645	22'826	23'599	24'992	25'434	25'476	25'549	25'604	25'709	25'723						
8	17'585	26'288	27'623	27'939	28'335	28'638	28'715	28'759	29'525	30'302							
9	17'419	25'941	27'066	27'761	28'043	28'477	28'721	28'878	28'948								
10	16'665	25'370	26'909	27'611	27'729	27'861	29'830	29'844									
11	15'471	23'745	25'117	26'378	26'971	27'396	27'480										
12	15'103	23'393	26'809	27'691	28'061	29'183											
13	14'540	22'642	23'571	24'127	24'210												
14	14'590	22'336	23'440	24'029													
15	13'967	21'515	22'603														
16	12'930	20'111															
17	12'539																
ϕ_j	−0.6700	−3.0000	−3.6900	−4.3600	−4.8200	−5.4700	−5.9000	−6.1000	−6.2000	−6.3000	−6.4000	−6.5500	−7.0000	−7.5000	−7.9700	−9.0000	
σ_j	0.0900	0.3600	0.6000	0.9000	1.1600	1.2900	1.3000	1.3100	1.3400	1.4000	1.5000	1.5000	1.3000	0.8000	0.2400	0.0400	
s_j	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	0.1980	

Table 5.1 Cumulative payments $C_{i,j} = \sum_{l=0}^{j} X_{i,l}$, $i + j \leq 17$, parameters ϕ_j, σ_j and s_j.

In Figure 5.1 we present the prior means ϕ_j, sample means $\overline{\xi}_j^{(t)}$ and posterior means $\phi_j^{(t)}$ based on the data $\mathcal{T}_t$ with $t = 17$. We see that the posterior mean smooths the sample mean using the prior mean with credibility weights $1 - \beta_j^{(t)}$; see also the credibility formula (3.2).

Next, we need to provide the term structure for the zero coupon bond prices at time $t = 17$ in order to calculate the best-estimate and the risk-adjusted reserves. We choose the actual CHF bond yield curve available from the Swiss National Bank. Finally, we choose the risk aversion parameters: $\alpha_1 = 0.02$ and $\alpha_2 = 1$. Now, we are ready to calculate the best-estimate and the risk-adjusted reserves, they are given in Table 5.2.

	$\mathcal{R}_{17}(\mathbf{X}_{(18)})$	$\mathcal{R}^+_{17}(\mathbf{X}_{(18)})$	$\mathrm{RM}_{17}(\mathbf{X}_{(18)})$
reserves under actual ZCB prices	23’977	25’066	1’089
nominal reserves, i.e. $P(17, k) \equiv 1$	24’672	25’814	1’142
discounting effect	695	748	53
discounting effect in %	2.82%	2.90%	4.64%

Table 5.2 Best-estimate reserves $\mathcal{R}_{17}(\mathbf{X}_{(18)})$, risk-adjusted reserves $\mathcal{R}^+_{17}(\mathbf{X}_{(18)})$ and risk margin $\mathrm{RM}_{17}(\mathbf{X}_{(18)})$ for the data set given in Table 5.1.

These reserves are calculated under the actual CHF bond yield curve and for nominal prices, i.e. $P(17, k) \equiv 1$. We observe that the discounting effect is quite small which comes from the fact that we are currently in a low interest rate period.

On the other hand we obtain a risk margin $\mathrm{RM}_{17}(\mathbf{X}_{(18)})$ of 1’089 which is 4.54% in terms of the best-estimate reserves $\mathcal{R}_{17}(\mathbf{X}_{(18)})$. Of course, the size of this risk margin heavily depends on the choice of the risk aversion parameters. In our case we have chosen these such that we obtain a similar risk margin as in the cost-of-capital approach under the parameter choices used for Solvency II. If we choose the split of total uncertainty approach from Salzmann and Wüthrich [10] with security loading $\phi = 2$ and cost-of-capital rate $c = 6\%$ (see formula (4.2) in [10] and TP.5.25 in [9]) we obtain for nominal reserves a risk margin of 1’107 (see also Table 5.4) which is comparable to the 1’142 of the probability distortion approach. Finally, the balancing between α_1 and α_2 was done such that if we turn off one of these two parameters then the risk margin has similar size; see Table 5.3. The question of the choice of the risk aversion parameters also needs input from the regulator. The latter gives the legal framework within which a loss portfolio transfer needs to take place. This question concerns whether or not the insurance portfolio is sent into run-off. Moreover, the regulator needs to decide at which state of the economy this transfer should take place between so-called willing financial agents because this also determines their risk aversion.

In Table 5.4 we compare the probability distortion approach (4.5) to the split of total uncertainty approach (proposed in Salzmann and Wüthrich [10]) and to the proportional

Swiss National Bank’s website: www.snb.ch

	$\mathcal{R}_{17}(\mathbf{X}_{(18)})$	$\mathcal{R}_{17}^{+}(\mathbf{X}_{(18)})$	$\mathrm{RM}_{17}(\mathbf{X}_{(18)})$
$\alpha_1 = 0.02$ and $\alpha_2 = 1$	23'977	25'066	1'089
$\alpha_1 = 0$ and $\alpha_2 = 1$	23'977	24'478	501
$\alpha_1 = 0.02$ and $\alpha_2 = 0$	23'977	24'546	568

Table 5.3 Best-estimate reserves $\mathcal{R}_{17}(\mathbf{X}_{(18)})$, risk-adjusted reserves $\mathcal{R}_{17}^{+}(\mathbf{X}_{(18)})$ and risk margin $\mathrm{RM}_{17}(\mathbf{X}_{(18)})$ for different risk aversion parameter choices.

nominal reserves	$\mathcal{R}_{17}(\mathbf{X}_{(18)})$	$\mathcal{R}_{17}^{+}(\mathbf{X}_{(18)})$	$\mathrm{RM}_{17}(\mathbf{X}_{(18)})$
probability distortion approach (4.5)	24'672	25'814	1'142
split of total uncertainty approach [10]	24'672	25'779	1'107
proportional scaling proxy TP.5.41 in [9]	24'672	25'350	678

Table 5.4 Comparison of probability distortion approach (4.5), split of total uncertainty approach [10] and proportional scaling proxy TP.5.41 in [9] in the risk measure framework of [10].

scaling proxy (which is the method used in QIS5 [9], Article TP.5.41, see also Salzmann and Wüthrich [10] and Keller [7]). We see that in this example the proportional scaling proxy is clearly below the other two approaches. This is further investigated in Figure 5.3 below (we also refer to Wüthrich [13]).

Next, we calculate the expected run-off of the best-estimate reserves and the risk margin. Therefore, we need a stochastic model for the development of the term structure which determines future zero coupon bond prices; see Theorem 4.5. For simplicity we only consider nominal cash flows for the run-off analysis which avoids modeling future zero coupon bond prices, i.e. we set $P(t,k) \equiv 1$ for $t, k \geq 17$. Figure 5.2 provides for this case the expected run-off of the best-estimate reserves and the risk margin.

Finally, we calculate the expected relative run-off of the risk margins defined by

$$w_k = \frac{\mathbb{E}\left[\mathrm{RM}_k\left(\mathbf{X}_{(k+1)}\right)\middle|\,\mathcal{T}_{17}, \mathcal{F}_{17}\right]}{\mathrm{RM}_{17}\left(\mathbf{X}_{(18)}\right)} \qquad \text{for } k \geq 17.$$

We observe that the split of total uncertainty approach $v_k(1)$, as defined in Salzmann and Wüthrich [10], gives a similar picture to the risk margin run-off pattern w_k, see Figure 5.3. On the other hand, the proportional scaling proxy $v_k(2)$ from Article TP.5.41 in QIS5 [9] (see also Salzmann and Wüthrich [10] and Keller [7]) clearly under-estimates run-off risks. This agrees with the findings in Wüthrich [13] and reflects that the expected claims reserves as volume measure for the run-off risk scaling is *not appropriate*. The main reason for this under-estimation of the proportional scaling proxy is that the payout of the claims reserves takes places much faster than the release of insurance-technical risk because we first settle small non-risky claims and risky claims stay in the

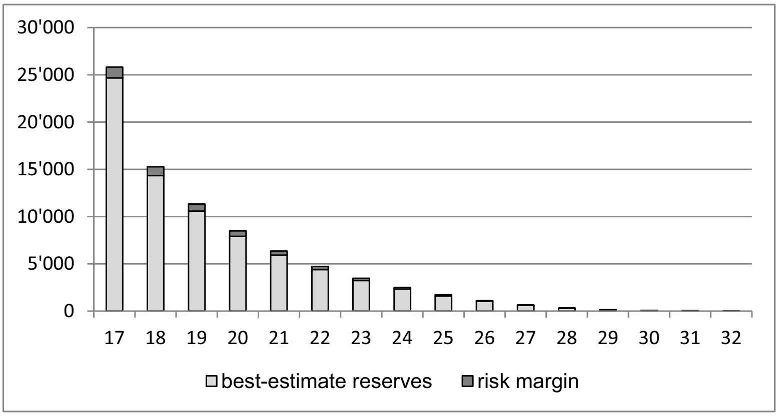

Figure 5.2 Expected run-off of the best-estimate reserves $\mathbb{E}\left[\mathcal{R}_k\left(\mathbf{X}_{(k+1)}\right)\middle|\,\mathcal{T}_{17}, \mathcal{F}_{17}\right]$ and the risk margin $\mathbb{E}\left[\mathrm{RM}_k\left(\mathbf{X}_{(k+1)}\right)\middle|\,\mathcal{T}_{17}, \mathcal{F}_{17}\right]$ for $k = 17, \ldots, n-1$.

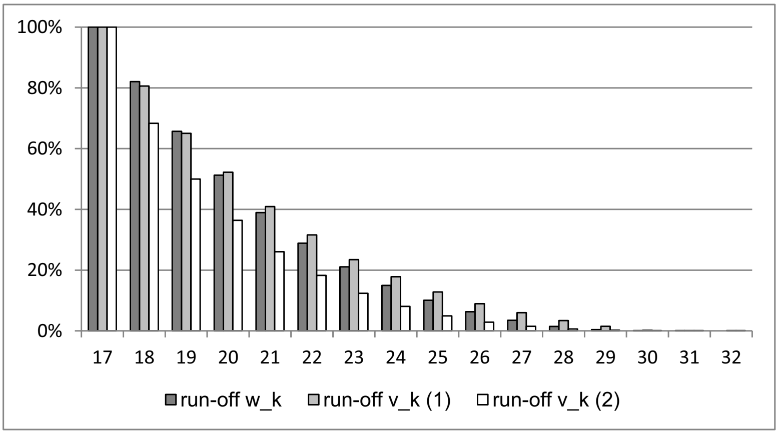

Figure 5.3 Expected relative run-off of the risk margins w_k, $k \geq 17$, compared to the split of total uncertainty approach $v_k(1)$ of Salzmann and Wüthrich [10] and the proportional scaling proxy $v_k(2)$ (see Article TP.5.41 in QIS5 [9] and Salzmann and Wüthrich [10]).

run-off portfolio for much longer accounting for the fact that their settlement is more difficult.

6 Conclusion

We have considered the concept of insurance-technical probability distortions for the calculation of the risk margin in non-life insurance. This concept is based on the assumption that financial agents are risk averse which is reflected by a positive correlation between the insurance-technical probability distortions and the insurance cash flows. This then provides, in a natural and mathematically consistent way, a positive risk margin. For our specific choice within the Bayesian log-normal chain ladder model we have found that this concept results in choosing prudent chain ladder factors. The prudence margin reflects the risk aversion in process risk and parameter uncertainty. We have compared our choice of the risk margin to the methods used in practice and we have found that the qualitative results are similar to the more advanced methods presented in Salzmann and Wüthrich [10].

In the present paper we have chosen one specific insurance-technical probability distortion because this choice has led to closed form solutions. Future research should investigate alternative constructions of probability distortions (according to market behavior of financial agents) and it should also investigate the question how these choices can be calibrated. In our example, we have assumed that the insurance cash flow is independent from financial market developments. This has resulted in the choice of the default-free zero coupon bond as replicating financial instrument. Future research should also analyze situations where this independence assumption is not appropriate.

A Proofs

Proof of Lemma 4.1: The strict positivity and the martingale property immediately follow from the definition of $\boldsymbol{\varphi}$. So there remains the proof of the normalization $\varphi_0 = 1$. Using the assumptions of Model 2.1 and the tower property we obtain (note that $\mathcal{T}_0 = \{\emptyset, \Omega\}$)

$$\begin{aligned}\varphi_0 &= \mathbb{E}\left[\varphi_n\right] = \mathbb{E}\left[\mathbb{E}\left[\varphi_n \middle| \boldsymbol{\Phi}\right]\right]\\ &= \mathbb{E}\left[\prod_{j=0}^{J-1} \exp\left\{(I\alpha_1+\alpha_2)\Phi_j - (I\alpha_1+\alpha_2)\phi_j - (I\alpha_1+\alpha_2)^2 s_j^2/2\right\}\right] = 1.\end{aligned}$$

This proves the claim. □

Proof of Theorem 4.2: Note that we have $C_{i,k-i} = X_{i,k-i} - X_{i,k-i-1}$, therefore it is sufficient to prove the claim for cumulative claims $C_{i,k-i}$. We first condition on the knowledge of the chain ladder parameters $\boldsymbol{\Phi}$,

$$\frac{1}{\varphi_t}\mathbb{E}\left[\varphi_k C_{i,k-i}\middle|\mathcal{T}_t\right] = \frac{1}{\varphi_t}\mathbb{E}\left[\varphi_n C_{i,k-i}\middle|\mathcal{T}_t\right] = \frac{1}{\varphi_t}\mathbb{E}\left[\mathbb{E}\left[\varphi_n C_{i,k-i}\middle|\mathcal{T}_t, \boldsymbol{\Phi}\right]\middle|\mathcal{T}_t\right].$$

Further,

$$\begin{aligned}\varphi_n &= \left[\prod_{j=1}^{J}\prod_{l=1}^{I}\exp\left\{\alpha_1 \xi_{l,j}\right\}\right]\\ &\quad\times \prod_{j=0}^{J-1}\exp\left\{\alpha_2\Phi_j - (I\alpha_1+\alpha_2)\phi_j - (I\alpha_1+\alpha_2)^2\frac{s_j^2}{2} - I\alpha_1^2\frac{\sigma_j^2}{2}\right\}.\end{aligned}$$

This means, that conditionally on $\mathbf{\Phi}$, the first term in the brackets is the only random term in φ_n. Define

$$\varphi_t^{\mathbf{\Phi}} = \mathbb{E}\left[\varphi_n \middle| \mathcal{T}_t, \mathbf{\Phi}\right] = \prod_{j=1}^{J} \prod_{l=1}^{(t-j)\wedge I} \exp\left\{\alpha_1 \xi_{l,j} - \alpha_1 \Phi_{j-1} - \alpha_1^2 \sigma_{j-1}^2/2\right\}$$
$$\times \prod_{j=0}^{J-1} \exp\left\{(I\alpha_1 + \alpha_2)\Phi_j - (I\alpha_1 + \alpha_2)\phi_j - (I\alpha_1 + \alpha_2)^2 \frac{s_j^2}{2}\right\}.$$

Hence, for $k > t$,

$$\mathbb{E}\left[\varphi_n C_{i,k-i} \middle| \mathcal{T}_t, \mathbf{\Phi}\right] = \mathbb{E}\left[\varphi_k^{\mathbf{\Phi}} C_{i,k-i} \middle| \mathcal{T}_t, \mathbf{\Phi}\right].$$

For the last term, note that $(\varphi_t^{\mathbf{\Phi}})_{t=0,\dots,n}$ is a martingale for the filtration $(\mathcal{T}_t, \mathbf{\Phi})_{t=0,\dots,n}$ and that the cumulative claim

$$C_{i,k-i} = C_{i,t-i} \prod_{j=t-i+1}^{k-i} \left(\exp\left\{\xi_{i,j}\right\} + 1\right)$$

only contains terms for accident year i which are conditionally independent given $\mathbf{\Phi}$. This implies that, for $k > t$,

$$\mathbb{E}\left[\varphi_k^{\mathbf{\Phi}} C_{i,k-i} \middle| \mathcal{T}_t, \mathbf{\Phi}\right] = \varphi_t^{\mathbf{\Phi}} C_{i,t-i} \prod_{j=t-i}^{k-i-1} \left(\exp\left\{\Phi_j + \alpha_1 \sigma_j^2 + \sigma_j^2/2\right\} + 1\right).$$

We therefore conclude that

$$\frac{1}{\varphi_t} \mathbb{E}\left[\varphi_k C_{i,k-i} \middle| \mathcal{T}_t\right] = \frac{C_{i,t-i}}{\varphi_t} \mathbb{E}\left[\varphi_t^{\mathbf{\Phi}} \prod_{j=t-i}^{k-i-1} \left(\exp\left\{\Phi_j + \alpha_1 \sigma_j^2 + \sigma_j^2/2\right\} + 1\right) \middle| \mathcal{T}_t\right]. \tag{A.1}$$

There are three important observations that allow to calculate this last expression. The first is that $\mathbb{E}\left[\varphi_t^{\mathbf{\Phi}} \middle| \mathcal{T}_t\right] = \varphi_t$ (which is the tower property for conditional expectations). The second comes from Theorem 3.1, namely we have posterior independence of the Φ_j's, conditionally given $\mathcal{T}_t$. This implies that expected values over the products of Φ_j can be rewritten as products over expected values. The third observation is that in the expected value of (A.1) we have exactly the same product terms as in φ_t except for the development periods $j \in \{t - i, \dots, k - i - 1\}$. This implies that all terms cancel except the ones that belong to these development parameters. If, in addition, we cancel all constants and $\mathcal{T}_t$-measurable terms we arrive at

$$\begin{aligned}&\frac{1}{\varphi_t}\mathbb{E}\left[\varphi_k C_{i,k-i}\middle|\mathcal{T}_t\right]\\&\quad= C_{i,t-i}\prod_{j=t-i}^{k-i-1}\Bigg(\mathbb{E}\Big[\exp\left\{([I-(t-j-1)]\alpha_1+\alpha_2)\Phi_j\right\}\\&\qquad\qquad\times\left(\exp\left\{\Phi_j+\alpha_1\sigma_j^2+\sigma_j^2/2\right\}+1\right)\mid\mathcal{T}_t\Big]\Bigg)\\&\qquad\qquad\times\left(\mathbb{E}\left[\exp\left\{([I-(t-j-1)]\alpha_1+\alpha_2)\Phi_j\right\}\mid\mathcal{T}_t\right]\right)^{-1}.\end{aligned}$$

So there remains the calculation of the terms in the product of the right-hand side of the equality above. Using Theorem 3.1 we obtain, for $j\in\{t-i,\dots,k-i-1\}$,

$$\begin{aligned}&\frac{\mathbb{E}\left[\exp\left\{([I-(t-j-1)]\alpha_1+\alpha_2)\Phi_j\right\}\left(\exp\left\{\Phi_j+\alpha_1\sigma_j^2+\sigma_j^2/2\right\}+1\right)\mid\mathcal{T}_t\right]}{\mathbb{E}\left[\exp\left\{([I-(t-j-1)]\alpha_1+\alpha_2)\Phi_j\right\}\mid\mathcal{T}_t\right]}\\&=\frac{\mathbb{E}\left[\exp\left\{(1+\alpha_2+[I-(t-j-1)]\alpha_1)\Phi_j\right\}\mid\mathcal{T}_t\right]}{\mathbb{E}\left[\exp\left\{(\alpha_2+[I-(t-j-1)]\alpha_1)\Phi_j\right\}\mid\mathcal{T}_t\right]}\exp\left\{\alpha_1\sigma_j^2+\sigma_j^2/2\right\}+1\\&=\exp\left\{\phi_j^{(t)}+\left(s_j^{(t)}\right)^2/2+\sigma_j^2/2\right\}\\&\qquad\times\exp\left\{(\alpha_2+[I-(t-j-1)]\alpha_1)\left(s_j^{(t)}\right)^2+\alpha_1\sigma_j^2\right\}+1.\end{aligned}$$

This proves Theorem 4.2. □

Proof of Theorem 4.5: We only prove the claim for the best-estimate reserves because the proof for the risk-adjusted reserves is completely analogous. From Corollary 3.2 we see that $\phi_l^{(t)}$ is the only random term in $f_l^{(t)}$. Therefore we can concentrate on this term. First we study the decoupling of $\phi_l^{(t)}$ conditionally given $\mathcal{T}_{t-1}$. If we use the credibility formula for this term we obtain

$$\phi_l^{(t)}=\beta_l^{(t)}\overline{\xi}_l^{(t)}+\left(1-\beta_l^{(t)}\right)\phi_l=\gamma_l^{(t-1)}\xi_{t-l-1,l+1}+\left(1-\gamma_l^{(t-1)}\right)\phi_l^{(t-1)},$$

with credibility weight given by

$$\gamma_l^{(t-1)}=\frac{s_l^2}{\sigma_l^2+(t-l-1)s_l^2}.$$

This is the well-known iterative update mechanism of credibility estimators; see for example Bühlmann and Gisler [3, Theorem 9.6]. Therefore, conditional on $\mathcal{T}_{t-1}$, $\xi_{t-l-1,l+1}$ is the only random term in $f_l^{(t)}$. Since all these terms belong to different accident years and development periods for $l \in \{t-i, \dots, J-1\}$ we have posterior independence, conditional on $\mathcal{T}_{t-1}$, which implies, for $k > t \geq I$, that

$$\begin{aligned}
&\mathbb{E}\left[C_{i,t-i} \prod_{l=t-i}^{j-2} f_l^{(t)} \left(f_{j-1}^{(t)} - 1 \right) \middle| \mathcal{T}_s \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[C_{i,t-i} \prod_{l=t-i}^{j-2} f_l^{(t)} \left(f_{j-1}^{(t)} - 1 \right) \middle| \mathcal{T}_{t-1} \right] \middle| \mathcal{T}_s \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[C_{i,t-i} \middle| \mathcal{T}_{t-1} \right] \prod_{l=t-i}^{j-2} \mathbb{E}\left[f_l^{(t)} \middle| \mathcal{T}_{t-1} \right] \mathbb{E}\left[f_{j-1}^{(t)} - 1 \middle| \mathcal{T}_{t-1} \right] \middle| \mathcal{T}_s \right] \\
&= \mathbb{E}\left[C_{i,t-i-1} \prod_{l=t-i-1}^{j-1} f_l^{(t-1)} \left(f_{j-1}^{(t-1)} - 1 \right) \middle| \mathcal{T}_s \right].
\end{aligned}$$

Iteration of this argument completes the proof. □

Acknowledgments. The authors would like to thank the referees for their constructive comments on an earlier version of the paper.

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